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Parabolic vortex equations and instantons of infinite energy

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Abstract

We study the vortex equations on parabolic bundles over a Riemann surface and prove a Hitchin–Kobayashi-type correspondence relating the existence of solutions to a certain stability condition. This is achieved by translating our problem into a four-dimensional one, via dimensional reduction arguments. In return we obtain examples of instantons of infinite energy.

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1. Introduction

Given a holomorphic vector bundle over a compact Riemann surface X there is a very natural condition for a Hermitian metric on it, namely that of being *projectively flat*. By a famous theorem of Narasimhan and Seshadri [NS] (see also [AB,D1]) the existence of such a metric is equivalent to the *stability* of the vector bundle (in the sense of Mumford). In higher dimensions projective flatness leads to an overdetermined equation and one needs to consider a weaker condition known as the *Hermitian–Einstein* condition (on a Riemann surface both conditions are equivalent). To write this equation one needs to choose a Kähler metric on the manifold. A Hermitian metric on the bundle is then called Hermitian–Einstein

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if the contraction of its curvature with the Kähler form is a constant multiple of the identity. The *Hitchin–Kobayashi* correspondence, proved by Donaldson, Uhlenbeck and Yau establishes that the existence of a Hermitian–Einstein metric is equivalent to the stability of the bundle—condition which in higher dimensions requires a Kähler metric in order to be defined.

The theorem of Narasimhan and Seshadri was generalized by Mehta and Seshadri [MS] (see also [Bi1,Bi2] for the formulation of this theorem that we shall use in this paper) to vector bundles with a *parabolic* structure. A parabolic structure on a vector bundle consists on a finite set of points on the Riemann surface, and filtrations by vector spaces of the fibres at the points, together with some real numbers called *weights*. Seshadri introduced a notion of stability for bundles with such structure. As in the usual case, this condition corresponds to the existence of a projectively flat metric, but now adapted in some precise sense to the parabolic structure, and whose connection is singular along the parabolic points. To define a parabolic structure in higher dimensions one replaces the points by smooth divisors and the filtrations by vector spaces by filtrations by vector bundles for the restriction of the vector bundle on the divisors. A Hitchin–Kobayashi correspondence for parabolic bundles over projective surfaces has been established by Biquard [Bi4].

In a different direction, a Hitchin–Kobayashi-type correspondence has been studied by Bradlow and García-Prada for *pairs* consisting of a vector bundle and a holomorphic section [B1,B2,GP1,GP2] and more generally for *triples* consisting of two vector bundles and a morphism between them [GP3,BGP]. In this case one defines a notion of stability which involves a real parameter analogous to the weights appearing in the parabolic structure. This condition is equivalent to the existence of a metric on the bundle satisfying, in the case of a pair, an equation known as the *vortex equation*, and in the case of a triple, a pair of equations—the so-called *coupled vortex equations*—for metrics on both bundles. These equations have terms involving the section and the parameter, in addition to the Hermitian–Einstein term.

In this paper we shall combine the above two types of additional structure to study the existence of metrics satisfying a generalization of the coupled vortex equations—that we shall call *parabolic vortex equations*—on two parabolic bundles over a Riemann surface linked by a map. We shall deal with two cases here. In the first one the map will be a morphism of parabolic bundles; we shall call these triples *parabolic triples*. In this case, the curvatures of the solutions are bounded. But for parabolic bundles it is natural to consider L^1 curvatures. As the equations have a linear term in the curvature and a quadratic term in the morphism, one can solve the problem with an L^2 morphism. This amounts to considering a meromorphic morphism with simple poles at the parabolic points, with the extra condition that the residues respect the parabolic structure strictly. The triples involved in this second case, which include of course the first ones, will be called *meromorphic parabolic triples*.

Putting both structures together we can define a notion of stability for meromorphic parabolic triples and show that this is equivalent to the existence of solutions to the parabolic vortex equations. The proof is based on a modification to the parabolic case of the dimensional reduction arguments, used in [BGP] to prove existence for the ordinary coupled vortex equations for a holomorphic triple.

The Hitchin–Kobayashi correspondence for triples is in fact closely related to the usual Hitchin–Kobayashi correspondence. Namely, to a holomorphic triple over a Riemann surface X one can canonically associate a holomorphic vector bundle over $X \times \mathbf{P}^1$. The stability of the triple turns out to be equivalent to the stability of this bundle. The parameter involved in the notion of stability for a triple is encoded in the Kähler metric on $X \times \mathbf{P}^1$, chosen to define stability. This vector bundle comes equipped with an action of $SU(2)$, lifted from that on $\mathbf{P}^1$. A Hermitian–Einstein metric is in fact $SU(2)$-invariant and is easily shown to be in correspondence with metrics on the bundles over X satisfying the coupled vortex equations.

In the case of a parabolic triple we can show that the corresponding bundle on $X \times \mathbf{P}^1$ admits a unique parabolic structure which is in fact $SU(2)$-invariant. As in the non-parabolic case the existence of an $SU(2)$-invariant Hermitian–Einstein metric adapted to the parabolic structure on the parabolic bundle over $X \times \mathbf{P}^1$ is equivalent to the existence of metrics adapted to the parabolic structures on the bundles over X satisfying the parabolic vortex equations. Our parabolic bundle over $X \times \mathbf{P}^1$ satisfies the hypotheses of the theorem of Biquard [Bi4] and hence the existence of a Hermitian–Einstein metric is equivalent to the parabolic stability of the bundle. In fact by $SU(2)$-invariance it is enough to check the stability condition for $SU(2)$-invariant subsheaves. We prove our theorem by showing that this parabolic $SU(2)$-invariant stability is equivalent to the parabolic stability of the parabolic triple.

The proof in the meromorphic case is a bit more involved, due to the fact that we cannot associate to the triple a parabolic bundle over $X \times \mathbf{P}^1$. Of course if the morphism is holomorphic (with no extra assumption) we can still associate a holomorphic bundle over $X \times \mathbf{P}^1$ to the triple, but this does not have a parabolic structure unless the morphism is a morphism of parabolic bundles, i.e. unless we are in the case considered above. However, in the meromorphic case outside of the parabolic points, the triple is holomorphic and we can still associate a holomorphic bundle over the non-compact manifold obtained from $X \times \mathbf{P}^1$ by removing the product of the set of parabolic points of X with $\mathbf{P}^1$. Doing some extra analysis one can apply a theorem of Simpson [Si] from which we deduce our theorem.

As a corollary of the proof we get examples of anti-self-dual connections on the complex surface $X \times \mathbf{P}^1$ minus a divisor, with non-trivial monodromy around the divisor and infinite energy (see Theorem 5.3). Although in the general situation finite energy anti-self-dual connections are related to parabolic bundles [Bi4], the question of whether infinite energy instantons correspond to some algebraic objects is still open.

The paper is organized as follows. In Section 2 we review the basic definitions of parabolic bundle and metric adapted to the parabolic structure. In Section 3 we introduce the meromorphic parabolic triples, a notion of stability for them, as well as the parabolic vortex equations and establish the correspondence theorem. In Section 4 we prove this theorem when the morphism is parabolic. In Section 5 we prove the theorem in the meromorphic case.

2. Preliminaries on parabolic bundles

We shall start by recalling some well-known facts about parabolic bundles.

Let X be a Riemann surface and S be a finite set of points of X. Let $E \to X$ be a holomorphic bundle.

Definition 2.1. A *parabolic structure* over S for the bundle E consists of a decreasing left-continuous filtration $(E_P^\alpha)_{0\leq\alpha\leq 1}$ of E_P such that $E_P^0 = E_P$ and $E_P^1 = \{0\}$ for each $P \in S$.

There are of course only finitely many jumps in the dimension of E_P^α— they occur at the *weights* of E at P. If α is a weight, its multiplicity is the dimension of $Gr^\alpha E_P = E_P^\alpha / E_P^{\alpha+\epsilon} (\epsilon > 0)$. Let

$$Gr\ E_P = \oplus_\alpha Gr^\alpha E_P$$

and α_P be the endomorphism of $Gr\ E_P$ acting by multiplication by α on $Gr^\alpha E_P$.

Remarks.

(1) Let $0 \leq \alpha_1 < \cdots < \alpha_n < 1$ be the weights of E at P. The parabolic structure can alternatively be defined by giving these weights and the filtration

$$E_P = E_P^{\alpha_1} \not\supseteq E_P^{\alpha_2} \not\supseteq \cdots \not\supseteq E_P^{\alpha_n} \not\supseteq 0\,.$$

(2) We will also need to consider the case when X is a complex surface. In this situation S will be a finite set of smooth disjoint divisors and the definition of a parabolic structure will be the same except that the E_P^α need to be replaced by holomorphic subbundles of $E|_D$, where $D \in S$.

Let E and E' be holomorphic bundles on X with parabolic structures over S. A morphism $\Phi : E \to E'$ is a holomorphic morphism such that if $P \in S$ and $0 \leq \alpha < 1$, then

$$\Phi_P(E_P^\alpha) \subset E_P'^\alpha\,.$$

The morphism is strict if for every $\epsilon > 0$

$$\Phi_P(E_P^\alpha) \subset E_P'^{\alpha+\epsilon}\,.$$

If $E' \subset E$ is a holomorphic subbundle, one obtains a parabolic structure over S for E' by writing

$$E_P'^\alpha = E'_P \cap E_P^\alpha,$$

and similarly for E/E'.

2.1. Adapted metrics

Let us fix a smooth metric on the Riemann surface X.

Theorem 2.2 ([Bi1,Bi3]). *There is an equivalence of categories between*
(1) *holomorphic bundles in X with parabolic structure on S and*
(2) *holomorphic Hermitian bundles on $X - S$ with L^p curvature for some $p > 1$, where the morphisms are holomorphic bounded morphisms on $X - S$.*

We shall describe briefly this correspondence.

Let us consider first a holomorphic parabolic bundle E. Let $P \in S$ and z be a local coordinate on X at P. Let (s_i) be a local basis of holomorphic sections of E such that E_P^α is generated by $(s_i(P))_{i \geq r - \dim E_P^\alpha + 1}$ ($r = \operatorname{rank} E$). Such a basis will be called an *adapted basis*. Let α_i be the weight such that $s_i(P) \in E_P^{\alpha_i}$, but $s_i(P) \notin E_P^{\alpha_i + \epsilon}$. We define (locally) a flat metric on E by

$$h_0(z) = \begin{pmatrix} |z|^{2\alpha_1} & & \\ & \ddots & \\ & & |z|^{2\alpha_r} \end{pmatrix}. \tag{1}$$

We do the same around each point of S and extend h_0 in a C^∞ manner over $X - S$. We get in this way a so-called *adapted metric* h_0 on $E|_{X-S}$, which will be the desired metric corresponding to E.

Conversely, let us consider a holomorphic Hermitian bundle $(E, |\cdot|)$ on $X - S$, with L^p curvature for some $p > 1$. One can show then that there is an extension of E to a unique parabolic bundle over X, such that

$$E_P^\alpha = \{s(P), s\colon \text{local holomorphic section with } |s(z)| = \mathrm{O}(|z|^\alpha)\}.$$

Furthermore, one has $s(P) \in E_P^\alpha$ and $s(P) \notin E_P^{\alpha+\epsilon}$ if and only if

$$|s(z)| \sim |z|^\alpha,$$

i.e. the two quotients are bounded.

Remark. Theorem 2.2 is true also for complex surfaces [Bi4], provided that the curvature is now in L^p for some $p > 2$, but the construction of an adapted metric is more complicated.

Proposition 2.3. *Let E be a holomorphic parabolic bundle on the compact Riemann surface X and h be an adapted metric on $X - S$ (i.e. h corresponds to E via Theorem 2.2), then*

$$-\frac{1}{2\pi \mathrm{i}} \int_{X-S} \operatorname{Tr}(F_h) = \deg(E) + \sum_{P \in S} \operatorname{Tr}(\alpha_P).$$

This number is called the parabolic degree of E and will be denoted by $\operatorname{pardeg}(E)$.

Remark. In the case of a compact complex Kähler surface (X, ω), one has the formula (for an adapted metric h)

$$\mathrm{pardeg}_{\omega}(E) = -\frac{1}{2\pi \mathrm{i}} \int\limits_{X-S} \mathrm{Tr}(F_h) \wedge \omega = \deg_{\omega}(E) + \sum_{D \in S} \mathrm{Tr}(\alpha_D)\langle [\omega], [D] \rangle .$$

2.2. Connections and gauge transformations

This section is special to Riemann surfaces. If h_0 is an adapted metric on a holomorphic parabolic bundle E and h is another adapted metric we can write

$$h(x, y) = h_0(gx, gy),$$

where g is an h_0-self-adjoint complex gauge transformation of $E|_{X-S}$.

The fact that the metric h is adapted with curvature L^p is equivalent to the fact that g must satisfy certain growth conditions of "weighted Sobolev" type. These conditions define a group $\mathcal{G}^p_{\mathbf{C}}$ of complex gauge transformations. There is also a space $\mathcal{A}^p$ of h_0-unitary connections with L^p curvature, locally asymptotic to the metric connection of (1), that is

$$d + \alpha_P \frac{\mathrm{d}z}{z} ,$$

or in the orthonormal basis $e_i = s_i / |z|^{\alpha_i}$,

$$d + \mathrm{i}\alpha_P \, \mathrm{d}\theta,$$

where $z = r \exp(\mathrm{i}\theta)$.

The group $\mathcal{G}^p_{\mathbf{C}}$ and its unitary subgroup $\mathcal{G}^p$ act on $\mathcal{A}^p$. If p is small enough $\mathcal{A}^p$, $\mathcal{G}^p$ and $\mathcal{G}^p_{\mathbf{C}}$ do not depend on any choice (in particular the choice of complementaries in the filtration) and they are therefore canonically attached to the parabolic structure. As in the usual case one has the two equivalent points of view:

(1) (adapted) metrics on a fixed holomorphic parabolic bundle,

(2) unitary connections (in $\mathcal{A}^p$) on a fixed Hermitian bundle.

3. Parabolic triples, vortex equations and stability

Let X be a compact Riemann surface and S be a finite set of points of X as above.

Definition 3.1. A *holomorphic triple* is a triple (E_1, E_2, Φ) consisting of two holomorphic vector bundles E_1 and E_2 over X and a morphism $\Phi : E_2 \to E_1$. If E_1 and E_2 have parabolic structures over S and Φ is a morphism of parabolic bundles the triple (E_1, E_2, Φ) is called a *parabolic triple*. If Φ is a meromorphic morphism with simple poles at the points of S, such that the residues respect the parabolic structure strictly, (E_1, E_2, Φ) is called a *meromorphic parabolic triple*.

Remark. Note that if the meromorphic morphism is in fact holomorphic, then the residue is zero and therefore it trivially respects the parabolic structure. So, in particular any parabolic triple is a meromorphic parabolic triple.

Let us consider a smooth metric on X with Kähler form ω. Let (E_1, E_2, Φ) be a meromorphic parabolic triple. We want to study the existence of Hermitian metrics h_1 and h_2 on E_1 and E_2, respectively, satisfying

$$\mathrm{i}\Lambda F_{h_1} + \Phi\Phi^* = 2\pi\tau_1 I_{E_1}, \qquad \mathrm{i}\Lambda F_{h_2} - \Phi^*\Phi = 2\pi\tau_2 I_{E_2}, \tag{2}$$

where Λ is contraction with the Kähler form. Φ^* is the adjoint of Φ with respect to h_1 and h_2, I_{E_1} and I_{E_2} are the identity endomorphisms of E_1 and E_2, respectively, and τ_1 and τ_2 are real parameters. Of course these equations are defined only on $X - S$.

The parameters τ_1 and τ_2 are not independent, but satisfy the linear relation

$$\tau_1 \operatorname{rank} E_1 + \tau_2 \operatorname{rank} E_2 = \operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2), \tag{3}$$

obtained by adding the traces of both equations and integrating (note that $\operatorname{Tr}\Phi\Phi^* = \operatorname{Tr}\Phi^*\Phi$).

When (E_1, E_2, Φ) is a holomorphic triple (i.e. E_1 and E_2 do not have parabolic structures) and h_1 and h_2 are ordinary Hermitian metrics, Eqs. (2) are the so-called *coupled vortex equations* introduced in [GP3,BGP]. We shall call our equations *parabolic vortex equations*.

As in the ordinary vortex equations, the existence of solutions to the parabolic ones is governed by a stability-type condition for the triple. To define this stability criterium we need to consider "subobjects" of a meromorphic parabolic triple. Let $T = (E_1, E_2, \Phi)$ be a meromorphic parabolic triple. A *subtriple* of T is a meromorphic parabolic triple $T' = (E_1', E_2', \Phi')$, where $E_1' \subset E_1$ and $E_2' \subset E_2$ are coherent subsheaves, and $\Phi' = \Phi|_{E_2'}$, i.e. we have the commutative diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\Phi} & E_1 \\ \uparrow & & \uparrow \\ E_2' & \xrightarrow{\Phi'} & E_1' \end{array}$$

If $E_1' = 0$ and $E_2' = 0$ or $E_1' = E_1$ and $E_2' = E_2$, the corresponding subtriples are called *trivial*.

Definition 3.2. Let σ be a real parameter. For any parabolic subtriple $T' = (E_1', E_2', \Phi')$ we define the parabolic σ-degree and parabolic σ-slope as

$$\operatorname{pardeg}_\sigma(T') = \operatorname{pardeg}(E_1' \oplus E_2') + \sigma \operatorname{rank} E_2', \tag{4}$$

and

$$\mu_\sigma(T') = \frac{\operatorname{pardeg}_\sigma(T')}{\operatorname{rank} E_1' + \operatorname{rank} E_2'}. \tag{5}$$

The meromorphic parabolic triple is called *σ-stable* if for all non-trivial subtriples $T' \subset T$, we have

$$\mu_\sigma(T') < \mu_\sigma(T). \qquad (6)$$

This generalizes the definition of stability for holomorphic triples given in [BGP].

Remarks.

(1) Given coherent subsheaves $E'_1 \subset E_1$ and $E'_2 \subset E_2$, they inherit, as explained in Section 2, parabolic structures, becoming parabolic subsheaves. On the other hand if $\Phi(E'_2) \subset E'_1$, taking $\Phi' = \Phi|_{E'_2}$, (E'_1, E'_2, Φ') becomes a parabolic subtriple. One can easily see that in order for T to be parabolic σ-stable it is enough to check (6) for the subtriples appearing in this way.

(2) As usual it will suffice to check (6) for saturated subtriples, i.e. those in which E'_1 and E'_2 are actually subbundles.

Definition 3.3. We say the triple $T = (E_1, E_2, \Phi)$ is *decomposable* if there are direct sum decompositions $E_1 = \bigoplus_{i=1}^n E_{1i}$, $E_2 = \bigoplus_{i=1}^n E_{2i}$, and $\Phi = \bigoplus_{i=1}^n \Phi_i$, such that $T_i = (E_{1i}, E_{2i}, \Phi_i)$ is a subtriple of T. We adopt the convention that if $E_{2i} = 0$ or $E_{1i} = 0$ for some i, then Φ_i is the zero map. We write $T = \bigoplus_{i=1}^n T_i$.

If T is not decomposable, we say T is *indecomposable*.

Our main objective in this paper will be to prove the following result.

Theorem 3.4. *Let $T = (E_1, E_2, \Phi)$ be an indecomposable meromorphic parabolic triple. Let τ_1 and τ_2 satisfy* (3), *and let $\sigma = \tau_1 - \tau_2$. Then E_1 and E_2 admit Hermitian metrics, adapted to the parabolic structures, satisfying* (2) *if and only if T is σ-stable.*

4. Proof of Theorem 3.4: Φ parabolic

We start by proving Theorem 3.4 when Φ is a holomorphic parabolic morphism. The general case is more involved and will be dealt with later on. Our proof is based on dimensional reduction methods. These consist, roughly speaking, in associating to a parabolic triple over X an equivariant parabolic bundle over $X \times \mathbf{P}^1$. The existence of solutions to Eqs. (2) translates into the existence of a Hermitian–Einstein metric on this bundle, for which we can apply a theorem of Biquard [Bi4]. The main ideas are an adaptation to the parabolic case of those used to prove existence of solutions to the coupled vortex equations for ordinary holomorphic triples (cf. [BGP]).

Proposition 4.1 ([BGP, Proposition 3.9]). *There is a one-to-one correspondence between holomorphic triples (E_1, E_2, Φ) over X and extensions over $X \times \mathbf{P}^1$ of the form*

$$0 \longrightarrow p^*E_1 \longrightarrow F \longrightarrow p^*E_2 \otimes q^*\mathcal{O}(2) \longrightarrow 0, \qquad (7)$$

where p and q are the projections from $X \times \mathbf{P}^1$ to the first and second factors, respectively.

Proof. The extensions over $X \times \mathbf{P}^1$ of the form (7) are parametrized by

$$H^1(X \times \mathbf{P}^1, p^*(E_1 \otimes E_2^*) \otimes q^*\mathcal{O}(-2));$$

but this is isomorphic to $H^0(X, E_1 \otimes E_2^*) \otimes H^1(\mathbf{P}^1, \mathcal{O}(-2)) \cong H^0(X, E_1 \otimes E_2^*)$, by means of the Künneth formula, the fact that $H^0(\mathbf{P}^1, \mathcal{O}(-2)) = 0$ and

$$H^1(\mathbf{P}^1, \mathcal{O}(-2)) \cong H^0(\mathbf{P}^1, \mathcal{O})^* \cong \mathbf{C}.$$

Therefore after fixing an element in $H^1(\mathbf{P}^1, \mathcal{O}(-2))$, the homomorphism Φ can be identified with the extension class defining F. □

Let $SU(2)$ act on $X \times \mathbf{P}^1$, trivially on X and in the standard way on $\mathbf{P}^1$. This action lifts to holomorphic actions on p^*E_1 and $p^*E_2 \otimes q^*\mathcal{O}(2)$, and since the action of $SU(2)$ on the extension class is trivial, one can lift the action to F, which becomes in this way an $SU(2)$-equivariant holomorphic bundle. If now E_1 and E_2 have parabolic structures it is natural to ask whether this equivariant bundle admits a parabolic structure or not. It turns out that the condition for this to happen is that Φ be a morphism of parabolic bundles, as one would naturally expect.

Let E be a holomorphic vector bundle over X, with a parabolic structure over a set of points $S \subset X$. It is clear that the parabolic structure on E induces a parabolic structure on p^*E over $D = S \times \mathbf{P}^1 \subset X \times \mathbf{P}^1$. The filtration for each $x \in S$

$$E\,|_x = F_1(E) \supset \ldots \supset F_n(E) \supset 0$$

induces the filtration

$$p^*E\,|_{D_x} = \mathcal{O}_{\mathbf{P}^1}^{d_1} \supset \ldots \supset \mathcal{O}_{\mathbf{P}^1}^{d_n} \supset 0,$$

where $D_x = \{x\} \times \mathbf{P}^1$, and $d_i = \dim F_i(E)$. We take for this filtration the same weights as the ones of the filtration of $E\,|_x$. Note that since D is $SU(2)$-invariant the parabolic structure on p^*E will be equivariant.

Let E_1 and E_2 be holomorphic vector bundles with a parabolic structure over S, then p^*E_1 and p^*E_2 have equivariant parabolic structures. Since tensoring with a line bundle preserves the parabolic structure, $p^*E_2 \otimes q^*\mathcal{O}(2)$ has also a parabolic structure, which is in fact $SU(2)$-equivariant since $q^*\mathcal{O}(2)$ is an $SU(2)$-equivariant line bundle. We want to characterize now the extensions (7) so that F admits a parabolic structure, inducing the starting parabolic structures on p^*E_1 and $p^*E_2 \otimes q^*\mathcal{O}(2)$. In contrast with what happens on a Riemann surface, in higher dimensions there is an obstruction for the existence of such parabolic structure. To explain this we shall digress for a moment to study the problem in more generality.

Let M be a compact complex manifold and D be an effective divisor. Let F_1 and F_2 be holomorphic vector bundles with a parabolic structure over D. We want to study the group $\mathrm{ParExt}(F_2, F_1)$ of parabolic extensions, i.e. the extensions

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0, \tag{8}$$

such that F has a parabolic structure over D inducing the starting parabolic structures on F_1 and F_2. Let $\mathcal{P}ar\mathcal{H}om(F_2, F_1)$ be the sheaf of parabolic morphisms from F_2 to F_1, i.e. the subsheaf of $\mathcal{H}om(F_2, F_1)$ defined for every open $U \subset M$ by

$$\mathcal{P}ar\mathcal{H}om(F_2, F_1)(U) = \{\text{parabolic morphisms } F_2|_U \longrightarrow F_1|_U\}.$$

Note that for every open set $U \subset M$, the parabolic structure over D of a bundle over M induces a parabolic structure over $D \cap U$ on the restriction of this bundle to U.

Lemma 4.2. $\mathrm{ParExt}(F_2, F_1) \cong H^1(\mathcal{P}ar\mathcal{H}om(F_2, F_1))$.

Proof. It is easy to see that the functor $\mathcal{P}ar\mathcal{H}om(F_2, \cdot)$ is exact (not only left exact) for parabolic bundles. This almost immediately implies that the group Ext^1 in the parabolic category is the above cohomology group.

More precisely, given an extension

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0,$$

one has the exact the sequence of sheaves

$$0 \longrightarrow \mathcal{P}ar\mathcal{H}om(F_2, F_1) \longrightarrow \mathcal{P}ar\mathcal{H}om(F_2, F) \longrightarrow \mathcal{P}ar\mathcal{H}om(F_2, F_2) \longrightarrow 0$$

and the image of the identity by the morphism

$$H^0(\mathcal{P}ar\mathcal{H}om(F_2, F_2)) \longrightarrow H^1(\mathcal{P}ar\mathcal{H}om(F_2, F_1))$$

classifies the extension. □

Let Q be the quotient of $\mathcal{H}om(F_2, F_1)$ by $\mathcal{P}ar\mathcal{H}om(F_2, F_1)$. It is clear that Q is a sheaf supported at D. Associated to

$$0 \longrightarrow \mathcal{P}ar\mathcal{H}om(F_2, F_1) \longrightarrow \mathcal{H}om(F_2, F_1) \longrightarrow Q \longrightarrow 0 \tag{9}$$

we have the long exact sequence

$$\begin{aligned} 0 \longrightarrow\ & H^0(\mathcal{P}ar\mathcal{H}om(F_2, F_1)) \longrightarrow H^0(\mathcal{H}om(F_2, F_1)) \longrightarrow H^0(Q) \\ \longrightarrow\ & H^1(\mathcal{P}ar\mathcal{H}om(F_2, F_1)) \longrightarrow H^1(\mathcal{H}om(F_2, F_1)) \longrightarrow H^1(Q) \longrightarrow . \end{aligned} \tag{10}$$

Thus, the obstruction for F in (8) to admit a parabolic structure is determined by an element of $H^1(Q)$. More precisely:

Proposition 4.3. *F admits a parabolic structure if the image of the extension* (8) *in $H^1(Q)$ is zero. The different parabolic structures that F might have are parametrized by the cokernel of $H^0(\mathcal{H}om(F_2, F_1)) \longrightarrow H^0(Q)$.*

Remark. If M is a Riemann surface, then Q is supported in a finite set of points and hence $H^1(Q) = 0$. Thus every extension F admits a parabolic structure (see [MS]).

Let us come back to our problem and study the group of parabolic extensions $\mathrm{ParExt}(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1)$.

Proposition 4.4. *$H^1(\mathcal{P}ar\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1))$ injects into $H^1(\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1))$, and is isomorphic to $H^0(X, \mathcal{P}ar\mathcal{H}om(E_2, E_1))$.*

Proof. We consider the short exact sequence (9) specialized to our situation. Q is supported at $D = S \times \mathbf{P}^1 \subset X \times \mathbf{P}^1$. Since $Q \subset \bigoplus \mathcal{H}om_{D_x}(\mathcal{O}(2)^{r_2}, \mathcal{O}^{r_1})$, where $D_x = \{x\} \times \mathbf{P}^1$, $r_1 = \operatorname{rank} E_1$ and $r_2 = \operatorname{rank} E_2$, then $H^0(Q) = 0$ and from (10) we obtain

$$0 \longrightarrow H^1(\mathcal{P}ar\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1)) \longrightarrow H^1(\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1)) \longrightarrow H^1(Q).$$

To prove the second assertion in the proposition it suffices to observe that by $SU(2)$-equivariance □

Lemma 4.5.

$$\mathcal{P}ar\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1) \cong p^*\mathcal{P}ar\mathcal{H}om(E_2, E_1) \otimes q^*\mathcal{H}om(\mathcal{O}(2), \mathcal{O}).$$

Hence applying the Künneth formula as in Proposition 4.1, we obtain

$$H^1(\mathcal{P}ar\mathcal{H}om(p^*E_2 \otimes q^*\mathcal{O}(2), p^*E_1)) \cong H^0(\mathcal{P}ar\mathcal{H}om(E_2, E_1)).$$

We have proved therefore that if (E_1, E_2, Φ) is a parabolic triple with Φ a morphism of parabolic bundles, the associated vector bundle over $X \times \mathbf{P}^1$ admits one, and only one, $SU(2)$-equivariant parabolic structure. We want to relate now the parabolic stability of (E_1, E_2, Φ) to the parabolic stability of this bundle. To do this we need to consider a Kähler metric on $X \times \mathbf{P}^1$. We shall take the product of $\frac{1}{2}\sigma$ times a metric on X with the Fubini–Study metric on $\mathbf{P}^1$. We shall normalize the volume of both X and $\mathbf{P}^1$ to one. The Kähler form of this metric depending on σ is

$$\omega_\sigma = \tfrac{1}{2}\sigma(p^*\omega_X) \oplus q^*\omega_{\mathbf{P}^1}. \tag{11}$$

Since the vector bundle F is $SU(2)$-equivariant we can consider the slightly weaker condition of *invariant stability*. This is like ordinary stability but the numerical condition on the slopes has to be satisfied only for $SU(2)$-invariant subsheaves of F.

Proposition 4.6. *Let $T = (E_1, E_2, \Phi)$ be a parabolic triple over X and let F be the parabolic bundle over $X \times \mathbf{P}^1$ associated to T. Then T is parabolic σ-stable if and only if F is $SU(2)$-invariant parabolic stable with respect to ω_σ.*

Proof. It is a straightforward generalization of the non-parabolic case (cf. [BGP]). We first note that for the statement to make sense we need $\sigma > 0$. This is not assumed in the definition of σ-stable but can in fact be obtained as a consequence of it. □

Lemma 4.7. *Let $T = (E_1, E_2, \Phi)$ be a σ-stable parabolic triple, then $\sigma > 0$.*

Proof. It is obtained by combining (6) for $T_1' = (\mathrm{Im}\,\Phi, 0, 0)$ and $T_2' = (E_1, \mathrm{Ker}\,\Phi, \Phi)$, and the observation that $\mathrm{Ker}\,\Phi$ and $\mathrm{Im}\,\Phi$ fit in the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\Phi \longrightarrow E_2 \longrightarrow \mathrm{Im}\,\Phi \longrightarrow 0. \qquad \square$$

Every subsheaf $F' \subset F$, and in particular every $SU(2)$-invariant subsheaf, inherits a parabolic structure from the parabolic structure of F. We want to prove that for every $SU(2)$-invariant subsheaf $F' \subset F$

$$\mu_\sigma(F') < \mu_\sigma(F)$$

is equivalent to the parabolic σ-stability of T, where μ_σ is the parabolic slope with respect to ω_σ. As shown in [BGP] there is a one-to-one correspondence between $SU(2)$-invariant subsheaves $F' \subset F$ and holomorphic subtriples $T' = (E_1', E_2', \Phi')$. F' is an extension which fits in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*E_1 & \longrightarrow & F & \longrightarrow & p^*E_2 \otimes q^*\mathcal{O}(2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & p^*E_1' & \longrightarrow & F' & \longrightarrow & p^*E_2' \otimes q^*\mathcal{O}(2) & \longrightarrow & 0. \end{array}$$

An easy calculation shows that

$$\begin{aligned} \mathrm{pardeg}_\sigma(F') &= \mathrm{pardeg}_\sigma(p^*E_1') + \mathrm{pardeg}_\sigma(p^*E_2' \otimes q^*\mathcal{O}(2)) \\ &= \mathrm{pardeg}\,E_1' + \mathrm{pardeg}\,E_2' + \sigma\,\mathrm{rank}\,E_2', \end{aligned}$$

since $\langle [\omega_\sigma], [D_x] \rangle = 1$, where $D_x = \{x\} \times \mathbf{P}^1$. Hence $\mu_\sigma(F') = \mu_\sigma(T')$ completing the proof.

Proof of Theorem 3.4 (parabolic case). The indecomposability of $T = (E_1, E_2, \Phi)$ is equivalent to the indecomposability, as an $SU(2)$-equivariant bundle, of the parabolic bundle F associated to T. By Proposition 4.6 the parabolic σ-stability of T is equivalent to the $SU(2)$-invariant parabolic stability of F with respect to ω_σ. We can now apply an equivariant version of the following theorem of Biquard.

Theorem 4.8 ([Bi4, Théorème 1.2]). *Let M be a projective surface, ω a Kähler form and $D \subset M$ a smooth divisor. Then a stable parabolic bundle over M admits a unique (up to a constant) adapted Hermitian–Einstein metric (the same is true with finitely many smooth disjoint divisors).*

Hence the $SU(2)$-invariant stability of F is equivalent to the existence of an $SU(2)$-invariant Hermitian–Einstein metric h on F adapted to the parabolic structure. This metric is of the form

$$h = p^*h_1 \oplus p^*h_2 \otimes q^*h_2'$$

for metrics h_1 and h_2 on E_1 and E_2, respectively, adapted to the parabolic structures, and h_2', an $SU(2)$-invariant metric on $\mathcal{O}(2)$ (see [GP3]) for similar statement in the non-parabolic case). Taking $\sigma = \tau_1 - \tau_2$ for τ_1 and τ_2 related by (3), one can see, using the same arguments of [BGP], that h is Hermitian–Einstein with respect to ω_σ if and only if h_1 and h_2 satisfy (2).

5. Proof of Theorem 3.4: Φ meromorphic

When Φ is meromorphic, the proof is more difficult, since there is no interpretation of the meromorphic triple (E_1, E_2, Φ) as a parabolic bundle over $X \times \mathbf{P}^1$. Instead of using [Bi4] we have to invoke directly Simpson's theorem [Si]. We begin by constructing an approximation to the solution.

Proposition 5.1. *Let (E_1, E_2, Φ) be a meromorphic parabolic triple. Then there exist adapted metrics h_1 and h_2 on E_1 and E_2 such that*

$$\mathrm{i}\Lambda F_{h_1} + \Phi\Phi^* \quad \textit{and} \quad \mathrm{i}\Lambda F_{h_2} - \Phi^*\Phi$$

are bounded on $X - S$.

Proof. It is a local problem near a marked point P with local coordinate z. Let (s_j) and (t_i) be local basis of holomorphic sections of E_1 and E_2, which are adapted to the parabolic structure. As a first approximation, one can look at the flat metrics in E_1 and E_2 as in (1),

$$H_0 = \begin{pmatrix} |z|^{2\alpha_1} & & \\ & \ddots & \\ & & |z|^{2\alpha_{r_1}} \end{pmatrix}, \qquad K_0 = \begin{pmatrix} |z|^{2\beta_1} & & \\ & \ddots & \\ & & |z|^{2\beta_{r_2}} \end{pmatrix},$$

where $r_1 = \operatorname{rank} E_1$ and $r_2 = \operatorname{rank} E_2$. In the orthonormal basis $e_j = s_j/|z|^{\alpha_j}$ and $f_i = t_i/|z|^{\beta_i}$, one has

$$\Phi = (\Phi_i^j |z|^{\beta_i - \alpha_j})_{i,j}$$

with:

- if $\beta_i > \alpha_j$ then Φ_i^j is meromorphic with simple pole at 0,
- if $\beta_i \le \alpha_j$ then Φ_i^j is holomorphic,

so that in any case $|\Phi| = \mathrm{O}(|z|^{-1+\epsilon})$.

Remark. In the case of a parabolic triple, one gets Φ_i^j holomorphic for $\beta_i \ge \alpha_j$ and Φ_i^j holomorphic with $\Phi_i^j(0) = 0$ for $\beta_i < \alpha_j$, so that Φ is bounded and H_0 and K_0 are the desired metrics.

Now we try to find $H = H_0 \mathrm{e}^u$ and $K = K_0 \mathrm{e}^v$ with u and v self-adjoint endomorphisms of E_1 and E_2, going to 0 when $z \to 0$. Write

$$\bar\partial_\alpha = \bar\partial - \frac{\alpha_P}{2}\frac{\mathrm{d}\bar z}{\bar z}, \qquad \partial_\alpha = \partial + \frac{\alpha_P}{2}\frac{\mathrm{d}z}{z},$$

we then want

$$Q_1(u,v) = \mathrm{i}\Lambda\bar\partial_\alpha(\mathrm{e}^{-u}\partial_\alpha \mathrm{e}^{u}) + \Phi \mathrm{e}^{-v}\Phi^* \mathrm{e}^{u},$$
$$Q_2(u,v) = \mathrm{i}\Lambda\bar\partial_\beta(\mathrm{e}^{-v}\partial_\beta \mathrm{e}^{v}) - \mathrm{e}^{-v}\Phi^* \mathrm{e}^{u}\Phi$$

to be bounded (Φ^* is the adjoint with respect to the fixed metrics H_0 and K_0). The principal terms in the linearizations of Q_1 and Q_2 are, respectively ($z = r\exp(\mathrm{i}\theta)$),

$$\mathrm{i}\Lambda\bar\partial_\alpha\partial_\alpha u = (-(r\partial_r)^2 - (\partial_\theta + \mathrm{i}\alpha_P)^2)u\frac{\Lambda \mathrm{i} dz \wedge d\bar z}{2r^2},$$
$$\mathrm{i}\Lambda\bar\partial_\beta\partial_\beta v = (-(r\partial_r)^2 - (\partial_\theta + \mathrm{i}\beta_P)^2)v\frac{\Lambda \mathrm{i}\, dz \wedge d\bar z}{2r^2}.$$

We use now the fact that the unit disk is conformally equivalent to the half-cylinder $\mathbf{R}_+ \times \mathbf{S}^1$, letting $s = -\ln r$. If α is a scalar, the operator

$$\Delta_\alpha = -(r\partial_r)^2 - (\partial_\theta + \mathrm{i}\alpha)^2 = -\partial_s^2 - (\partial_\theta + \mathrm{i}\alpha)^2$$

on the whole cylinder $\mathbf{R}\times\mathbf{S}^1$ is an isomorphism between weighted Hölder spaces (see [MP]),

$$\Delta_\alpha : C_\delta^{2+\eta} \to C_\delta^\eta,$$

if $\delta + \alpha \notin \mathbf{Z}$ and $\delta - \alpha \notin \mathbf{Z}$ (the space C_δ^η is the space of functions f such that $\mathrm{e}^{\delta s} f \in C^\eta$).

If $g \in C_\delta^\eta(\mathbf{R}_+ \times \mathbf{S}^1)$, one can extend smoothly g in $C_\delta^\eta(\mathbf{R} \times \mathbf{S}^1)$ with compact support in $\mathbf{R}_- \times \mathbf{S}^1$ and solve $\Delta_\alpha f = g$ with $f \in C_\delta^{2+\eta}(\mathbf{R} \times \mathbf{S}^1)$ or at least $f \in C_{\delta'}^{2+\eta}(\mathbf{R} \times \mathbf{S}^1)$ for all $\delta' < \delta$ if δ is critical (i.e. $\delta + \alpha$ or $\delta - \alpha \in \mathbf{Z}$). Restricting f to $\mathbf{R}_+ \times \mathbf{S}^1$, we get a right inverse for Δ_α, sending $C_\delta^\eta(\mathbf{R}_+ \times \mathbf{S}^1)$ to $C_\delta^{2+\eta}(\mathbf{R}_+ \times \mathbf{S}^1)$ or at least $C_{\delta'}^{2+\eta}(\mathbf{R}_+ \times \mathbf{S}^1)$ for all $\delta' < \delta$.

Call ζ the C^∞ function $\Lambda \mathrm{i}\, dz \wedge \frac{1}{2} d\bar z$. We use the previous considerations to solve

$$(-\partial_s^2 - (\partial_\theta + \mathrm{i}\alpha_P)^2)u_0 = -\frac{r^2}{\zeta}\Phi\Phi^*, \qquad (-\partial_s^2 - (\partial_\theta + \mathrm{i}\beta_P)^2)v_0 = -\frac{r^2}{\zeta}\Phi^*\Phi.$$

The most singular terms in $\Phi\Phi^*$ and $\Phi^*\Phi$ are equivalent to $r^{-2+\delta}$ ($\delta > 0$), so one can find u_0 and v_0 in $C_\delta^{2+\eta}$ (perhaps in $C_{\delta'}^{2+\eta}$ for all $\delta' < \delta$, but we will from now on forget these details). Then $Q_1(u_0, v_0)$ and $Q_2(u_0, v_0)$ are now in $C_{\delta+\epsilon_0}^{2+\eta}$ instead of $C_\delta^{2+\eta}$ for some $\epsilon_0 > 0$.

The solutions u and v are obtained by a finite process of iteration, as finite sums

$$u = u_0 + u_1 + \cdots, \qquad v = v_0 + v_1 + \cdots$$

To write the next step we look at $u = u_0 + u_1$ and $v = v_0 + v_1$ and write the principal terms of Q_1 and Q_2, using the equations satisfied by u_0 and v_0:

$$\mathrm{i}\Lambda\bar\partial_\alpha\partial_\alpha u_1 + \text{terms in } u_0, v_0 \quad \text{with weight } \delta + \epsilon_0,$$
$$\mathrm{i}\Lambda\bar\partial_\beta\partial_\beta v_1 + \text{terms in } u_0, v_0 \quad \text{with weight } \delta + \epsilon_0.$$

Therefore one can find u_1 and v_1 in $C_{\delta+\epsilon_0}^{2+\eta}$ such that both terms are zero. Hence $Q_1(u,v)$ and $Q_2(u,v)$ are in $C_{\delta+\epsilon_0+\epsilon_1}^\eta$.

At each step, we get a weight ϵ_i which is easily seen to be bounded from below in terms of the weights α_j and β_i, so, in a finite number of steps, we obtain $Q_1(u, v)$ and $Q_2(u, v)$ bounded.

Proof of Theorem 3.4. In the case of a meromorphic parabolic triple we shall also use dimensional reduction, although in an analytic way, since we do not have any longer an algebraic interpretation on $X \times \mathbf{P}^1$. So we shall consider the open manifold $(X - S) \times \mathbf{P}^1$ with the Kähler form ω_σ. On this open manifold we have a holomorphic bundle F given by

$$0 \longrightarrow p^*E_1 \longrightarrow F \longrightarrow p^*E_2 \otimes q^*\mathcal{O}(2) \longrightarrow 0,$$

which is now defined by $\Phi|_{X-S}$.

By Proposition 5.1, we choose metrics h_1 and h_2 on E_1 and E_2 such that $\mathrm{i}\Lambda F_{h_1} + \Phi\Phi^*$ and $\mathrm{i}\Lambda F_{h_2} + \Phi^*\Phi$ are bounded. These metrics induce a metric h on F such that
(1) $F_h \in L^p (p > 1)$,
(2) ΛF_h is bounded,
(3) h is $SU(2)$-equivariant.

Lemma 5.2. *The $SU(2)$-equivariant L^2_1 subsheaves of F over $(X - S) \times \mathbf{P}^1$ are in 1–1 correspondence with subtriples of (E_1, E_2, Φ) over X (with the same degree).*

Proof. An $SU(2)$-equivariant L^2_1 subsheaf of F over $(X - S) \times \mathbf{P}^1$ induces an L^2_1 subtriple of (E_1, E_2, Φ) over $X - S$. The L^2_1 condition implies (see [UY]) that we have holomorphic subbundles over $X - S$ and that these subbundles extend over X [Si, Lemma 10.6] with the right degree [Si, Lemma 10.5]. □

We deduce from Lemma 5.2 that F is equivariantly analytically stable over $(X - S) \times \mathbf{P}^1$. We can now apply Simpson's theorem [Si, Theorem 1] to obtain a (unique) Hermitian–Einstein metric k on F, mutually bounded with h and $SU(2)$-equivariant ((1) and (2) are necessary in order to use Simpson's theorem).

Hence as in [BGP] we get over $X - S$ two metrics k_1 and k_2 on E_1 and E_2, mutually bounded with h_1 and h_2, such that

$$\mathrm{i}\Lambda F_{k_1} + \Phi\Phi^* = 2\pi\tau_1 I_{E_1}, \qquad \mathrm{i}\Lambda F_{k_2} - \Phi^*\Phi = 2\pi\tau_2 I_{E_2}.$$

To finish we have to prove that these metrics are adapted to the parabolic structures.

Since they are mutually bounded with h_1 and h_2, we deduce that $\Phi\Phi^*$ and $\Phi^*\Phi$ are in L^p for some $p > 1$ and (from the equations) so are F_{k_1} and F_{k_2}. By Theorem 2.2, the metrics k_1 and k_2 are then adapted to E_1 and E_2 (their curvatures are is L^p and they are mutually bounded with adapted metrics).

In the process of proving Theorem 3.4 we have obtained the following.

Theorem 5.3. *There is a one-to-one correspondence between*
(1) irreducible $SU(2)$-equivariant anti-self-dual connections over $(X - S) \times \mathbf{P}^1$ with L^p-curvature, for some $p > 1$ (modulo gauge transformations),
and

(2) *stable meromorphic parabolic triples over $X - S$ (modulo isomorphisms).*

Moreover, the anti-self-dual connection has finite energy if and only if in the associated triple (E_1, E_2, Φ) Φ is a parabolic morphism, i.e. $\Phi \in H^0(\mathcal{P}ar\mathcal{H}om(E_2, E_1))$.

Remark. This result provides with lots of examples of instantons of infinite energy.

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